8 Optimal Detection for Additive Noise Channels: 1-D Case

We now derive the optimal demodulator for the waveform channel. From the previous chapter, we have seen that instead of analyzing the waveform channel, we can convert it to an equivalent vector channel. The length of the vector is the same as the size K of the orthonormal basis for the waveforms $s_1(t), s_2(t), \ldots, s_M(t)$. In this chapter, we will assume $K = 1$. This is the case, for example, when we use PAM.

Definition 8.1. **Detection Problem**: When $K = 1$, our problem under consideration is simply that of **detecting** the scalar message S in the presence of additive noise N. The received signal R is given by

$$R = S + N.$$

- S is selected from an alphabet $\mathcal{S}$ containing M possible values $s^{(1)}$, $s^{(2)}$, …, $s^{(M)}$.
- $p_S\left(s^{(j)}\right) = P\left[S = s^{(j)}\right] \equiv p_j$.
- S and N are independent.

A detector's job is to guess the value of the channel input S from the value of the received channel output R. We denote this guessed value by $\hat{S}$. An optimal detector is the one that minimizes the (symbol) error probability $P(\mathcal{E}) = P\left[\hat{S} \neq S\right]$.

8.2. The analysis here is very similar to what we have done in Chapter 3. Here, for clarity, we note some important differences:

- In Chapter 3, The channel input and output are denoted by X and Y, respectively. Here, they are denoted by S and R.
- In Chapter 3, the transition probabilities are arbitrary and summarized by the matrix $\mathbf{Q}$. Here, the transition probabilities is basically controlled by the additive noise.
- In Chapter 3, both X and Y are discrete. Here, S is discrete. However, because noise is continuous, R will be a continuous random variable.

Even with these differences, several techniques that we used in Chapter 3 will be applicable here.

Example 8.3. Review: To re-connect with what we studied in Chapter 3, let's try to find the **Q** matrix when the additive noise is discrete. Suppose

the pmf for the message $p_1 = 0.3$ $\hat{s}^{(1)}$ $\hat{s}^{(2)}$ p_2

$$p_S(s) = \begin{cases} 0.3, & s = -1, \\ 0.7, & s = 1, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and } p_N(n) = \begin{cases} 0.2, & n \in \{-0.5, +0.5\}, \\ 0.6, & n = 0, \\ 0, & \text{otherwise}. \end{cases}$$

pmf for noise

Because $R = S + N$, we know that

[315 Sec 9.2]

(a) given $S = -1$, we have $R = -1 + N$:

$$p_{R|S}(r|-1) = \begin{cases} 0.2, & r \in \{-1.5, -0.5\}, \\ 0.6, & r = -1, \\ 0, & \text{otherwise}. \end{cases}$$

$$\equiv$$

$$P[R = r | S = -1] = P[N = r - (-1)] = p_N(r - (-1))$$

n	$p_N(n)$	$-1+n$	$1+n$
-0.5	0.2	-1.5	0.5
0	0.6	-1	1
0.5	0.2	-0.5	1.5

(b) given $S = 1$, we have $R = 1 + N$:

$$p_{R|S}(r|1) = \begin{cases} 0.2, & r \in \{0.5, 1.5\}, \\ 0.6, & r = 1, \\ 0, & \text{otherwise}. \end{cases}$$

$$= P[N = r - 1] = p_N(r - 1)$$

The **Q** matrix is given by

$$\mathbf{Q} = \Big[\, P[R = r | S = s] \,\Big] = \begin{array}{c|c} s \backslash r & \begin{matrix} -1.5 & -1 & -0.5 & 0.5 & 1 & 1.5 \end{matrix} \\ \begin{matrix} -1 \\ 1 \end{matrix} & \begin{bmatrix} 0.2 & 0.6 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.6 & 0.2 \end{bmatrix} \end{array} \xrightarrow[\times p_2 \; 0.7]{\times p_1 \; 0.3} \begin{bmatrix} .06 & .18 & .06 & 0 & 0 & 0 \\ 0 & 0 & 0 & .14 & .42 & .14 \end{bmatrix}$$

$$\hat{s}_{ML}(r) = \begin{cases} -1, & r \in \{-1.5, -1, -0.5\} \\ 1, & r \in \{0.5, 1, 1.5\} \end{cases} \qquad \hat{s}_{MAP}(r) = \hat{s}_{ML}(r) =$$

Note that each row of the **Q** matrix is simply a shifted copy of the noise pmf. The amount of shift is the corresponding value of s for that row.

8.4. Formula-wise, when the additive noise is discrete, each row of the **Q** matrix (as in Example 8.3) is given by

$$p_{R|S}(r|s) = p_N(r - s). \tag{47}$$

analog

8.5. When the additive noise is continuous, there are uncountably many possible values for the channel output R. Hence, representing conditional probabilities in the form of a matrix **Q** does not make sense here.

R = S+N

When R is continuous, the conditional pmf $p_{R|S}(r|s)$ is replaced by the conditional pdf $f_{R|S}(r|s)$. For additive noise N with pdf $f_N(n)$, we have

$$f_{R|S}(r|s) = f_N(r-s). \tag{48}$$

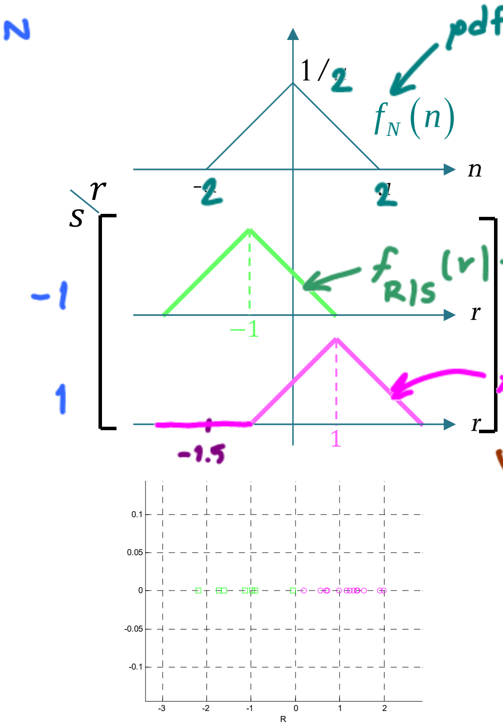
Example 8.6. Suppose the discrete additive noise in Example 8.3 is replaced by a continuous additive noise:

$$\int_{-\infty}^{\infty} f_N(n)\,dn = 1$$

area $= \frac{1}{2} \times 2a \times \frac{1}{a} = 1$

Here, set $a = 2$.

S	R = S+N
1	1.3800
-1	-1.7267
1	1.5429
1	1.9886
1	1.1556
1	1.8981
1	0.9856
1	0.5699
1	0.7173
-1	-0.9756
1	1.2335
-1	-1.1320
-1	-0.0612
-1	-2.1877
-1	-0.9035
1	0.6967
1	1.3127
1	0.1829
1	1.4026
-1	-1.6058



This replaces the "Q matrix".

Figure 40: Binary PAM under "Triangular" Noise

8.7. The optimal detector, which minimizes the error probability, is the **MAP detector**:

⇒ "P matrix"

corresponding s value is the answer

"Q matrix"

$$\hat{s}_{\text{MAP}}(r) = \arg\max_{s\in\mathcal{S}} p_S(s) f_{R|S}(r|s) = \arg\max_{s\in\mathcal{S}} p_S(s) f_N(r-s) \tag{49}$$

max in each column

Because event $[W=j]$ is the same as event $[S=s^{(j)}]$, we also have

$$\hat{w}_{\text{MAP}}(r) = \arg\max_{j\in\{1,2,\ldots,M\}} p_j f_N\left(r - s^{(j)}\right). \tag{50}$$

When the prior probabilities are ignored, we have the (sub-optimal) **ML detector**:

$$\hat{s}_{\text{ML}}(r) = \arg\max_{s\in\mathcal{S}} f_{R|S}(r|s) = \arg\max_{s\in\mathcal{S}} f_N(r-s). \tag{51}$$

and

$$\hat{w}_{\mathrm{ML}}(r) = \underset{j \in \{1,2,\ldots,M\}}{\arg\max}\, f_N\left(r - s^{(j)}\right). \tag{52}$$

8.8. Graphically, here are the steps to find the MAP detector:

(a) Plot $p_1 f_N(r - s^{(1)})$, $p_2 f_N(r - s^{(2)})$, $\ldots$, $p_M f_N(r - s^{(M)})$.

- Note that they are functions of r.
- This is similar to scaling the rows of the $\mathbf{Q}$ matrix by the corresponding prior probabilities in Chapter 3 to get the $\mathbf{P}$ matrix.

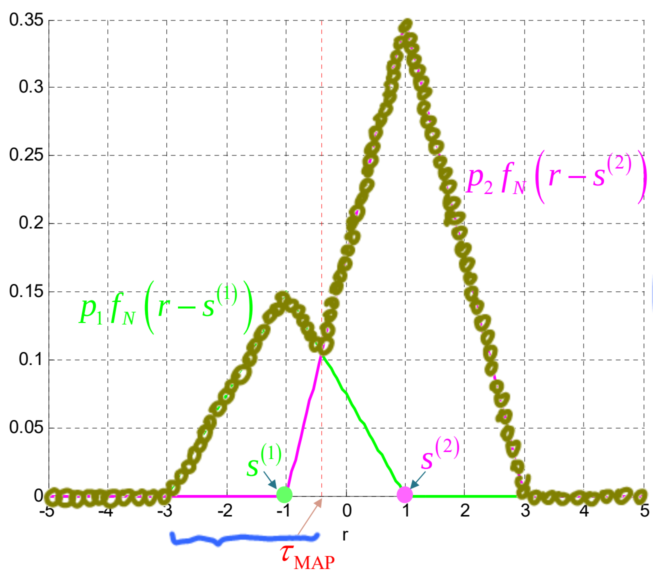
(b) Select the maximum plot for each (observed) r value.

- If there are multiple max values, select any.
- The corresponding $s^{(j)}$ is the value of $\hat{s}_{\mathrm{MAP}}$ at r.

Example 8.9. Back to Example 8.6.

$$\hat{s}_{\mathrm{MAP}}(r) = \begin{cases} -1, & -3 < r < \tau_{\mathrm{MAP}} \\ 1, & \tau_{\mathrm{MAP}} < r < 3 \\ \text{any}, & \text{otherwise} \end{cases}$$

-0.4



$$\hat{s}_{\mathrm{MAP}}(r) = \begin{cases} s^{(1)}, & r < \tau_{\mathrm{MAP}}, \\ s^{(2)}, & r \geq \tau_{\mathrm{MAP}}. \end{cases}$$

$$\hat{w}_{\mathrm{MAP}}(r) = \begin{cases} 1, & r < \tau_{\mathrm{MAP}}, \\ 2, & r \geq \tau_{\mathrm{MAP}}. \end{cases}$$

$$\hat{s}_{\mathrm{MAP}}(r) = \begin{cases} \overset{s^{(1)}}{-1}, & r < \tau_{\mathrm{MAP}}, \\ \underset{s^{(2)}}{1}, & r \geq \tau_{\mathrm{MAP}}. \end{cases}$$

Figure 41: MAPD for Binary PAM under "Triangular" Noise

Definition 8.10. The ith **decision "region"**, denoted by $\mathcal{D}_i$ for a ~~decoder~~ detector $\hat{s}(r)$ is defined as the collection of all the r values at which r is decoded as $s^{(i)}$.

$\mathcal{D}_1 = (-\infty, \tau_{\mathrm{MAP}})$

$\mathcal{D}_2 = [\tau_{\mathrm{MAP}}, \infty)$

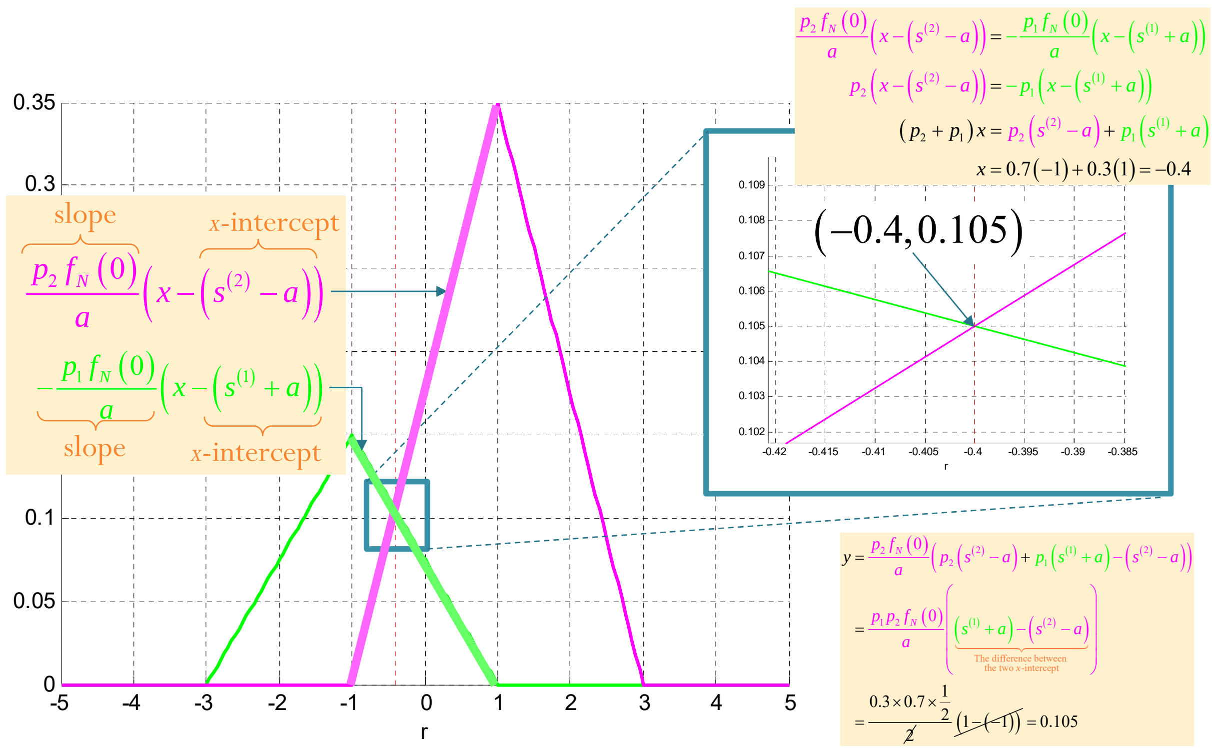


Figure 42: Solving for τ_{MAP} in MAPD for Binary PAM under "Triangular" Noise

- The collection $\mathcal{D}_1$, $\mathcal{D}_2$, $\ldots$, $\mathcal{D}_M$ should partition the whole observable values (support) of R.

Example 8.11. Back to Example 8.6.

$$\hat{s}_{MAP}(r)=\begin{cases} s^{(1)}, & r\in\mathcal{D}_1, \\ s^{(2)}, & r\in\mathcal{D}_2, \end{cases} \quad \text{where} \quad \begin{aligned} \mathcal{D}_1 &= (-\infty, \tau_{\text{MAP}}), \\ \mathcal{D}_2 &= [\tau_{\text{MAP}}, \infty). \end{aligned}$$

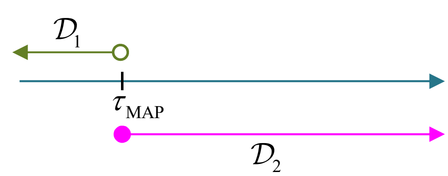


Figure 43: Decision Regions in MAPD for Binary PAM under "Triangular" Noise

8.12. The error probability of a detector can be found via its success probability

$$
\begin{aligned}
P(\mathcal{C}) &= \sum_{i=1}^{M} P\left(\mathcal{C}|S=s^{(i)}\right)P\left[S=s^{(i)}\right] = \sum_{i=1}^{M} P\left[R\in D_i \middle| S=s^{(i)}\right]p_i \\
&= \sum_{i=1}^{M} p_i P\left[S+N\in \mathcal{D}_i \middle| S=s^{(i)}\right] = \sum_{i=1}^{M} p_i P\left[N+s^{(i)}\in \mathcal{D}_i\right] \\
&= \sum_{i=1}^{M} p_i \int_{\mathcal{D}_i} f_N\left(r-s^{(i)}\right)dr = \sum_{i=1}^{M} \int_{\mathcal{D}_i} p_i f_N\left(r-s^{(i)}\right)dr.
\end{aligned}
$$

This gives

$$
\begin{aligned}
P(\mathcal{E}) &= 1-P(\mathcal{C}) \\
&= \sum_{i=1}^{M} p_i \int_{D_i^c} f_N\left(r-s^{(i)}\right)dr = \sum_{i=1}^{M} \int_{D_i^c} p_i f_N\left(r-s^{(i)}\right)dr.
\end{aligned}
$$

Although, at first, the above expressions may look complicated, it is similar to what we did in Chapter 3: graphically, the area under the max (selected) plot is $P(\mathcal{C})$.

$P =$

$P(\mathcal{E}) = 1 - P(\mathcal{C})$

$P(\mathcal{C})$ = sum of the chosen values

Example 8.13. Back to Example 8.6.

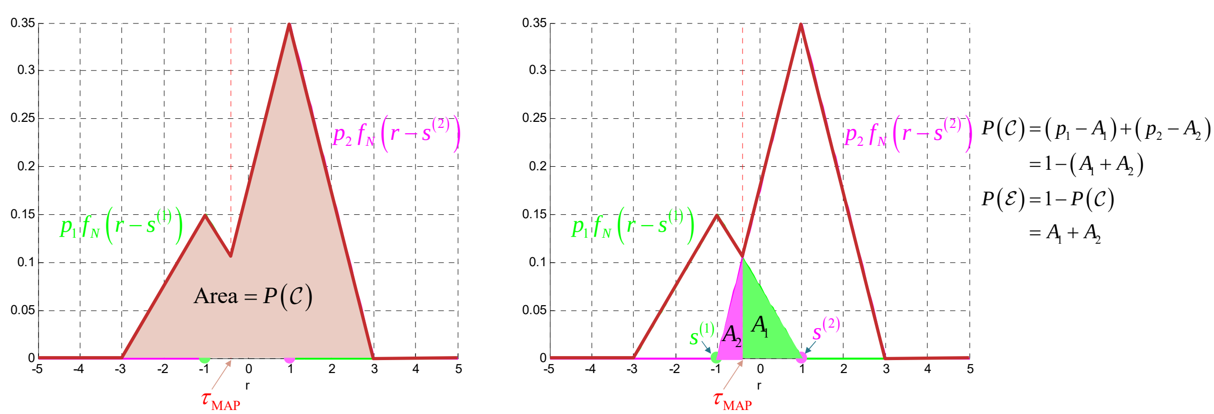
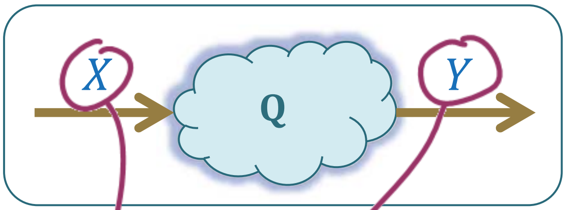


Figure 44: Probability of Successful Detection for Binary PAM under “Triangular” Noise

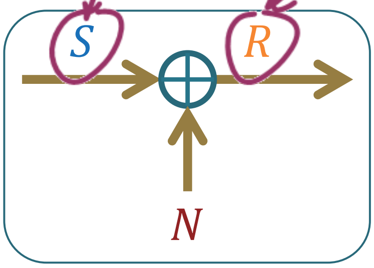
Model: CH3 vs. CH8

- Chapter 3:



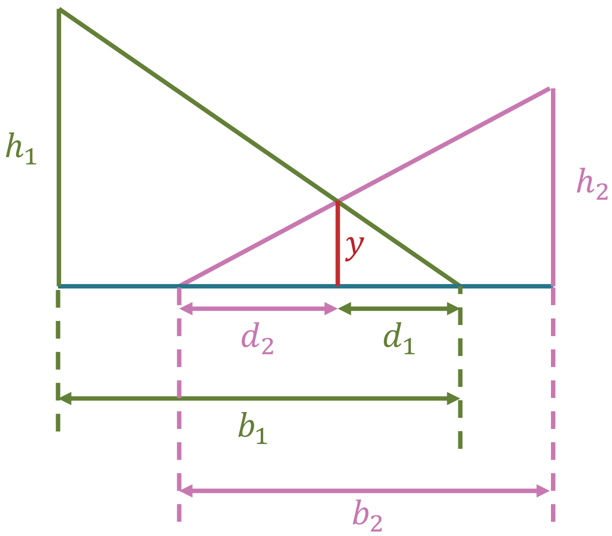
$$\mathbf{Q} = \begin{array}{c|c} {}_{x}\diagdown^{y} & \begin{matrix} y_1 & \cdots & y_j & \cdots & y_n \end{matrix} \\ \begin{matrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_M \end{matrix} & \begin{bmatrix} \ddots & & \cdot^{\cdot^{\cdot}} \\ & Q(y_j|x_i) & \\ \cdot^{\cdot^{\cdot}} & & \ddots \end{bmatrix} \end{array}$$

- Chapter 8: Additive Model: $R = S + N$



- N is discrete with pmf $p_N(n)$
  - Intermediate model
  - Can still use Q matrix to characterize the conditional probability of the output given the input.
- N is continuous with pdf $f_N(n)$
  - Need to use conditional density to characterize the conditional probability of the output given the input.

Similar Triangles



Similar Triangles

- $\dfrac{h_1}{b_1} = \dfrac{y}{d_1}$
- $\dfrac{h_2}{b_2} = \dfrac{y}{d_2}$

ECS 452: In-Class Exercise # 24

Instructions

1. Working alone is always permitted. However, working in groups is also allowed if social distancing can be used (via, e.g., online group chat/call). For group work, **the group cannot be the same as any of your former group after the midterm.**
2. Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: 28 / 4 / 2020			
Name	ID (last 3 digits)		

In a binary antipodal signaling scheme, the message S is randomly selected from the alphabet set $\mathcal{S} = \{-2, 2\}$ with $p_1 = P[S = -2] = 0.6$ and $p_2 = P[S = 2] = 0.4$. The message is corrupted by an independent additive noise N. The detector observes $R = S + N$ at its input.

a. Suppose that the noise is a *discrete* random variable whose pmf is $p_N(n) = \begin{cases} 0.2, & n \in \{-2, 2\}, \\ 0.6, & n = 0, \\ 0, & \text{otherwise.} \end{cases}$

i. List all possible values of R.

When $S = -2$, we have $R = S + N = -2 + N$. N can be $-2,0,2$; so, R can be $-4,-2,0$.
When $S = 2$, we have $R = S + N = 2 + N$. N can be $-2,0,2$; so, R can be $0,2,4$.
Combining the two cases, R can be $-4,-2,0,2,4$.

ii. Find the corresponding $\mathbf{Q}$ matrix. (Recall that the $\mathbf{Q}$ matrix contains $P[R = r | S = s]$.)

Recall that $P[R = r | S = s] = p_N(r - s)$.

We first find the values of $r - s$.

$$\begin{array}{c|c} s \backslash r & \begin{matrix} -4 & -2 & 0 & 2 & 4 \end{matrix} \\ \begin{matrix} -2 \\ 2 \end{matrix} & \begin{bmatrix} -2 & 0 & 2 & 4 & 6 \\ -6 & -4 & -2 & 0 & 2 \end{bmatrix} \end{array}$$

Then, we calculate $p_N(r - s)$ from the provided expression.

$$\begin{array}{c|c} s \backslash r & \begin{matrix} -4 & -2 & 0 & 2 & 4 \end{matrix} \\ \begin{matrix} -2 \\ 2 \end{matrix} & \begin{bmatrix} 0.2 & 0.6 & 0.2 & 0 & 0 \\ 0 & 0 & 0.2 & 0.6 & 0.2 \end{bmatrix} \end{array}$$

iii. Find the MAP detector $\hat{s}_{\text{MAP}}(r)$.

When the noise is a discrete RV, we follow the technique from [3.40] in CH3. Starting with the $\mathbf{Q}$ matrix, we scale its rows by the prior probabilities to get the $\mathbf{P}$ matrix.

$$\begin{array}{c|c} s \backslash r & \begin{matrix} -4 & -2 & 0 & 2 & 4 \end{matrix} \\ \begin{matrix} -2 \\ 2 \end{matrix} & \begin{bmatrix} 0.2 & 0.6 & 0.2 & 0 & 0 \\ 0 & 0 & 0.2 & 0.6 & 0.2 \end{bmatrix} \end{array} \begin{matrix} \xrightarrow{\times 0.6} \\ \xrightarrow{\times 0.4} \end{matrix} \begin{array}{c|c} s \backslash r & \begin{matrix} -4 & -2 & 0 & 2 & 4 \end{matrix} \\ \begin{matrix} -2 \\ 2 \end{matrix} & \begin{bmatrix} 0.12 & 0.36 & 0.12 & 0 & 0 \\ 0 & 0 & 0.08 & 0.24 & 0.08 \end{bmatrix} \end{array}$$

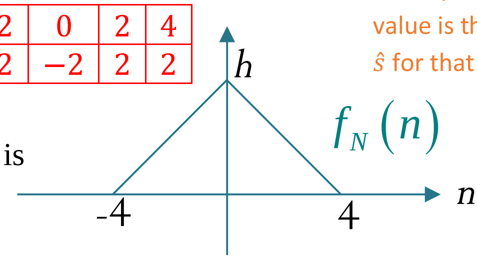
Then, we select the max. value in each column. The corresponding s value is the value of $\hat{s}$ for that r.

$$\hat{s}_{\text{MAP}}(r) = \begin{cases} -2, & r \in \{-4, -2, 0\}, \\ 2, & r \in \{2, 4\}. \end{cases}$$

or, in tabular form:

r	-4	-2	0	2	4
$\hat{s}_{\text{MAP}}(r)$	-2	-2	-2	2	2

b. Suppose that the noise is a *continuous* random variable whose pdf is



i. What is the value of h?

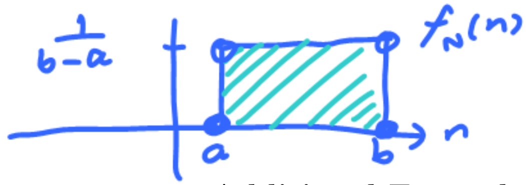
Area must be 1. Here, we have $\frac{1}{2} \times h \times 8 = 1$. Therefore, $h = \frac{1}{4}$.

ii. Suppose $R = 0$ is observed. Find the corresponding output of the MAP detector.

$$p_1 f_N\left(r - s^{(1)}\right) = 0.6 f_N(0 - (-2)) = 0.6 \times f_N(2) = 0.6 \times \frac{h}{2} = 0.3h = 0.075$$

$$p_2 f_N\left(r - s^{(2)}\right) = 0.4 f_N(0 - 2) = 0.4 \times f_N(-2) = 0.4 \times \frac{h}{2} = 0.2h = 0.050$$

Because $p_1 f_N\left(r - s^{(1)}\right) > p_2 f_N\left(r - s^{(2)}\right)$ at $r = 0$, we conclude that $\hat{s}_{\text{MAP}}(0) = s^{(1)} = -2$.



Problem 16. In a binary antipodal signaling scheme, the message S is randomly selected from the alphabet set $\mathcal{S} = \{-3, 3\}$ with $p_1 = P[S = -3] = 0.3$ and $p_2 = P[S = 3] = 0.7$. The message is corrupted by an independent additive noise N which is uniform on $[-4, 4]$.

(a) Find the pdf of the noise.

Recall that the uniform pdf on $[a,b]$ is given by $f_N(n) = \begin{cases} \frac{1}{b-a}, & a < n < b, \\ 0, & \text{otherwise.} \end{cases}$

[EES315]

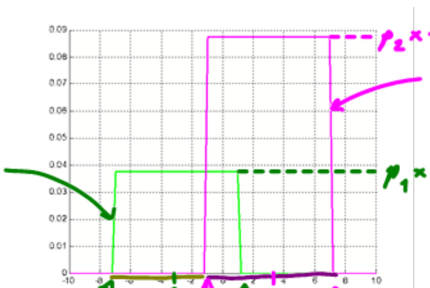
You may recall, from EES315, that the pdf of a continuous RV is not unique. One can modify a few (actually, countably many points) and still have the pdf for the same RV. So, here, it does not matter whether we have strict inequalities or not.

Here, $a = 4$ and $b = -4$. So, $f_N(n) = \begin{cases} 1/8, & -4 < n < 4, \\ 0 & \text{otherwise.} \end{cases}$

(b) Find the MAP detector $\hat{s}_{\text{MAP}}(r)$.

The MAP detector is given by $\hat{s}_{MAP}(r) = \arg\max_{s} p_s f_N(r - s)$. This is true regardless of the pdf of the noise.

Here, there are two possible values for s: -3 and 3. So we compare $\underset{0.3}{p_1} f_N(r - \underset{-3}{s^{(1)}})$ and $\underset{0.7}{p_2} f_N(r - \underset{3}{s^{(2)}})$.



Observe that

① when $-7 < r < -1$,

$$p_1 f_N(r - s^{(1)}) > p_2 f_N(r - s^{(2)}).$$

Therefore, $\hat{s}_{MAP}(r) = s^{(1)} = -3$ in this region.

② when $-1 < r < 7$,

$$p_1 f_N(r - s^{(1)}) < p_2 f_N(r - s^{(2)}).$$

Therefore, $\hat{s}_{MAP}(r) = s^{(2)} = 3$ in this region.

③ when $r < -7$ or $r > 7$, the pdf in both cases are 0.

So, these are the impossible regions. The received signal R won't fall in these regions. Therefore, it does not matter how the detector behaves in this region.

④ From part (a), because we have some flexibility in defining the values of the noise pdf at n = a,b, the comparison results at the jump points r = -7, -1, 7 in this part may give different answers. However, begin continuous random variable, the probabilities that the reveived signal R will be exactly any one of these three points is zero. Therefore, we can define $\hat{s}_{MAP}(r)$ to be anything here as well.

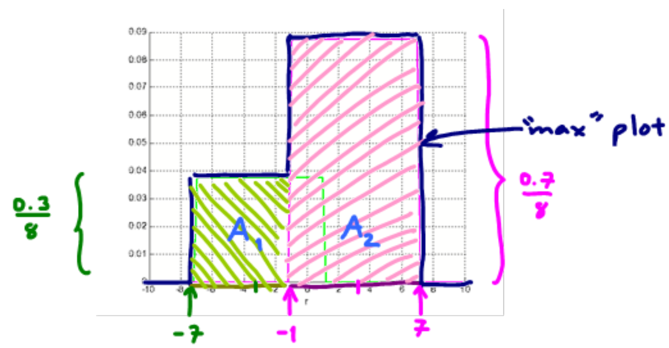
Conslusion: Combining the results from the four parts above, we have

$$\hat{s}_{MAP}(r) = \begin{cases} -3, & -7 < r < -1, \\ 3, & -1 < r < 7, \\ \text{anything}, & \text{otherwise} \end{cases} = \begin{cases} -3, & r < -1 \\ 3, & r \geq -1 \end{cases}$$

To further simplify the expression, we choose the "anything" parts above such that they can be combined into adjacent intervals.

(c) Evaluate the error probability of the MAP detector.

Recall that for MAP detector, $P(\mathcal{C})$ = area under the "max" plot.



Area $A_1 = \frac{0.3}{8}(-1-(-7)) = \frac{0.3 \times 6}{8} = 0.225$

Area $A_2 = \frac{0.7}{8}(7-(-1)) = \frac{0.7 \times 8}{8} = 0.7$

So, $P(\mathcal{C}) = 0.925$

Therefore, $P(\mathcal{E}) = 1 - P(\mathcal{C}) = 1 - 0.925 = 0.075$

Problem 17. In a binary antipodal signaling scheme, the message S is randomly selected from the alphabet set $\mathcal{S} = \{-3, 3\}$ with $p_1 = P[S=-3] = 0.3$ and $p_2 = P[S=3] = 0.7$. The message is corrupted by an independent additive exponential noise N whose pdf is

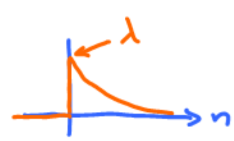
$$f_N(n) = \begin{cases} \frac{1}{2}e^{-n/2}, & n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the MAP detector $\hat{s}_{\text{MAP}}(r)$.

Some facts about exponential noise:

In this question, $\lambda = \frac{1}{2}$

① $f_N(n) = \begin{cases} \lambda e^{-\lambda n}, & n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$



② $P[N > n] = \int_n^\infty f_N(n)\,dn = \begin{cases} e^{-\lambda n}, & n \geq 0 \\ 1, & n < 0 \end{cases}$ ⇒

The whole area under this graph is p because any pdf has unit area.